# What is $\psi(x,t)$ ?

The prob. of finding the particle in space $d^3r = |\psi(r,t)|^2 d^3r$

Probability density function

$$\int_{\text{all space}} |\psi(r,t)|^2 d^3r = 1$$

$$\int \psi^* \psi \, d^3r = 1$$

The average or expectation value of an observable associated with operator A is :

$$\boxed{\langle A \rangle = \int_{-\infty}^{\infty} \psi^* \hat{A} \, \psi \, d^3r}$$

Operator

Example:

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x\, \psi(x,t)\, dx = \int_{-\infty}^{\infty} x\, |\psi(x,t)|^2$$

Momentum operator in QM is defined as : $p = -i\hbar\nabla$

or in 1D : $P_x = -i\hbar\dfrac{\partial}{\partial x}$

Example:

So for averge of $P_x$ we have:

$$\langle P_x \rangle = \int_{-\infty}^{\infty} \psi^*(x)\,(-i\hbar)\frac{\partial}{\partial x}\psi(x)\,dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \psi^*(x)\,\frac{\partial \psi(x)}{\partial x}\,dx$$

Show that $E$ is the energy:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = C\psi$$

$$-\frac{\hbar^2}{2m}\int \psi^*\frac{d^2\psi}{dx^2}\,dx + \int \psi^* V \psi\,dx = \int C\overbrace{\psi^*\psi\,dx}^{=1}$$

$$\frac{1}{2m}\int \psi^*\underbrace{\left(i\hbar\frac{d}{dx}\right)}_{P_x}\underbrace{\left(i\hbar\frac{d}{dx}\right)}_{P_x}\psi\,dx + \langle V\rangle = C$$

$$\underbrace{\frac{1}{2m}\langle P_x^2\rangle}_{\langle T\rangle} + \langle V\rangle = C$$

$$\langle T\rangle + \langle V\rangle = E \quad \checkmark$$

It is also possible to show that:

$$m\frac{d\langle x\rangle}{dt} = \langle P_x\rangle \quad \text{(Exercise)}$$

(hint: write down $\langle x\rangle$, take the derivative, use the Schrodinger equ & integrate by part)

Example: what is the wave function for an electron in free space

$$\psi(r,t) = \underset{\downarrow \atop ?}{\psi(r)}\, e^{-i\omega t}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r) + \underbrace{V(r)}_{=0}\psi(r) = E\psi(r)$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

$$\psi'' = -\frac{2mE}{\hbar^2}\psi$$

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \Rightarrow \frac{2mE}{\hbar^2} = k^2$$

$$\psi'' = -k^2 \psi$$

$$\psi(r) = A e^{ik \cdot r} + B e^{-ik \cdot r}$$

$$\psi(r,t) = (\underset{\leftarrow}{A e^{ik \cdot r}} + \underset{\rightarrow}{B e^{-ik \cdot r}}) e^{-i\omega t}$$

For an electron moving left and 1D:

$$\Psi(x,t) = A e^{ikx} e^{-i\omega t}$$

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Operator form:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r) + V \Psi(r) = E \Psi(r)$$

$$\underbrace{\left(-\frac{\hbar^2}{2m} \nabla^2 + V\right)}_{H \text{ (hamiltonian)}} \psi(r) = E \Psi(r)$$

$$\hat{H} \psi(r) = E \psi(r)$$

eigenfunction   eigenvalue $\rightarrow$ multiple values

$$\hat{H} \Psi_n(r) = E_n \Psi_n(r)$$